Copulas for Random Closed Sets

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Abstract

Let φ be a Choquet-capacity functional on $K = (K_1, K_2)$ and each K_i of which T_i is its capacity functional. Then there exists a unique sub-copula C such that (1) $Dom(C) = Ran(T_1) \times Ran(T_2)$, (2) For all $(K_1, K_2) \in \mathcal{K}$, $C(T_1(K_1), T_2(K_2)) = \varphi(K_1 \times K_2)$. In this research, we have the theorems to construction a unique sub-copula C which we can apply to any other random closed set, for example, random (vertice) graph.

1 Introduction

The standard definition of a copula is a multivariate distribution function defined on the unit cube $[0, 1]^n$, with uniformly distributed marginals. This definition is very natural if one considers how a copula is derived from a continuous multivariate distribution function; indeed in this case the copula is simply the original multivariate distribution function with transformed univariate margins. Alternatingly, we denote by $DomH$ and $RanH$ the domain and range respectively of H. Furthermore, a function f will be called nondecreasing whenever $x \leq y$ implies that $f(x) \leq f(y)$. A statement about points of a set $S \subseteq R^n$, where S is typically the real line or the unit cube $[0, 1]^n$, is said to hold almost everywhere if the set of points of S where the statement fails to hold has Lebesgue measure zero.[B.Nelson;1999]

Definition 1.1. A real function H of n variables is n-increasing if $V_H(B) \geq 0$ for all n-boxes B whose vertices lie in $DomH$.

Definition 1.2. An n-dimensional copula is a function C with domain $[0, 1]^n$ such that

(1) C is grounded and n-increasing.

(2) C has margins $C_k, k = 1, 2, ..., n$, which satisfy $C_k(u) = u$ for all u in $[0, 1]$.

The following theorem is known as Sklar's Theorem. It is the most important result regarding copulas, and is used in essentially all applications of copulas.

***Theorem* 1.3.** *Let H be an n-dimensional distribution function with margins $F_1, ..., F_n$. Then there exists an n-copula C such that for all x in $\bar{R}^n$,*

$$H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)).$$

If $F_1, ..., F_n$ are all continuous, then C is unique; otherwise C is uniquely determined on $RanF_1 \times ... \times RanF_n$. Conversely, if C is an n-copula and $F_1, ..., F_n$ are distribution functions, then the function H defined above is an n-dimensional distribution function with margins $F_1, ..., F_n$.

For the proof, see [Sklar ;1996].

***Corollary* 1.4.** *Let H be an n-dimensional distribution function with continuous margins $F_1, ..., F_n$ and copula C. Then for any u in $[0, 1]^n$,*

$$C(u_1, ..., u_n) = H(F^{-1}(u_1), ..., F^{-1}(u_n))$$

Consider the functions M_n, Π_n and W_n defined on $[0, 1]^n$ as follows:

$$M_n(u) = min\{u_1, ..., u_n\},$$

$$\Pi_n(u) = u_1...u_n,$$

$$W_n(u) = max\{u_1 + \cdot\cdot\cdot + u_n - n + 1, 0\}.$$

The functions M_n and Π_n are n-copulas for all $n \geq 2$ whereas the function W_n is not a copula for any $n \geq 3$

Let $X_1, ..., X_n$ be random variables with continuous distribution functions $F_1, ..., F_n$, respectively, and joint distribution function H. Then $(X_1, ..., X_n)$ has a unique copula C, where C is given by Sklar's theorem. The standard copula representation of the distribution of the random vector $(X_1, ..., X_n)$ then becomes:

$$\begin{aligned} H(x_1, ..., x_n) &= \mathcal{P}(X_1 \leq x_1, ..., X_n \leq x_n) \\ &= C(F_1(x_1), ..., F_n(x_n)). \end{aligned}$$

Since $X_1, ..., X_n$ are independent if and only if $H(x_1, ..., x_n) = F_1(x_1)...F_n(x_n)$ for all $x_1, ..., x_n$ in R, the result is the following.

***Theorem* 1.5.** *Let $(X_1, ..., X_n)$ be a vector of continuous random variables with copula C, then $X_1, ..., X_n$ are independent if and only if $C = \Pi_n$.*

Copulas provide a natural way to study and measure dependence between random variables. Copula properties are invariant under strictly increasing transformations of the underlying random variables. Linear correlation is most frequently used in practice as a measure of dependence. However, since linear correlation is not a copula-based measure of dependence, it can often be quite misleading and should not be taken as the canonical dependence measure. [Paul Embrechts, Filip Lindskog and Alexander McNeil; 2001]. In this research, copulas are a tool for modeling and capturing the dependence of random graphs.

By those three different correlation structures and the concept of copula, we can generate the joint distribution function for random variables $P_{i1}, P_{i2}, ..., P_{in}$ via the fact that the edge probabilities in our random graphs are all random variables in the closed interval [0,1]. Yiyi Shi described some idea of this joint distribution function by the Gaussian copula as the following:

$$F(p_{i1}, p_{i2}, ..., p_{in}) = \mathcal{P}(P_{i1} \leq p_{i1}, ..., P_{in} \leq p_{in})$$
$$= C(p_{i1}, p_{i2}, ...p_{in}) = H_{\rho_i}(\Phi(p_{i1}), \Phi(p_{i2}), ..., \Phi(p_{in}))$$
$$= \frac{1}{(2\pi)^{n/2}} \int \cdots \int exp(-\frac{1}{2}\mathbf{x}^T \Sigma^{-1}\mathbf{x})d\mathbf{x}$$

where $\Phi^{-1}(x)$ is the inverse of the standard normal distribution function $\Phi(x)$

$H_{\rho_i}(\mathbf{x})$ is the multi-variate standard normal distribution function with correlation matrix Σ having entries all ρ_i's except for diagonals being 1's. But, in his research did not verify this copula in form of the function from $[0,1] \times [0,1] \to [0,1]$. Moreover, it is observed by simulations that the correlation effect would cause a non-Gaussian degree fluctuation, especially when the correlation is large.[Y. Shi; 2009]

2 Preliminaries

A copula is a multivariate distribution with all univariate marginal distributions being uniformly distributed on the unit interval,$[0,1]$; hence C is the distribution of a multivariate uniform random vector. For a bivariate distribution F with margins F_1 and F_2, the copula associated with F is a distribution function C : $[0,1]^2 \to [0,1]$ that satisfies for $(x,y) \in \mathbb{R}$,

$$F(x,y) = C(F_1(x), F_2(y))$$

The copula C is uniquely determined on the unit square whenever F_1 and F_2 are continuous. The copula itself characterises the dependence between the random variables X and Y with marginal distributions F_1 and F_2 . Thus the copula representation resolves the joint distribution into the marginals F_1 and F_2 and the dependence structure C. When X and Y are discrete random variables taking values on some lattice, Ω , the copula, C, is unique provided $(x,y) \in \Omega$ but not elsewhere; this non-uniqueness is of no consequence however since the region outside Ω is not of interest in the discrete case (NELSEN, 2006). The representation and uniqueness follows essentially from a multivariate extension to the probability integral transformation (JOE, 1997).

To define a copula, one should note that for any arbitrary increasing F_1 and F_2, if u and v are uniform on the interval $[0,1]$, then $x = F_1^{-1}(u)$ and $y = F_2^{-1}(v)$ are distributed according to F_1 and F_2. The mapping from u to x and v to y are one-to-one when x, y are continuous and many-to-one when x, y are discrete. The multivariate function C on the unit cube $[0,1]^2$

$$C(u,v) = F(F_1^{-1}(u), F_2^{-1}(v))$$

is called a copula if it is a continuous distribution function and each marginal is a uniform distribution on $[0,1]$. It defines a correspondence between the marginal distributions F_1, F_2 and the joint distribution F. Although many types of copulas exist, Gaussian copulas are a natural choice when moving beyond the bivariate case.

For any copula, $C(u,v) \leq C(u,1) = u$ and $C(u,v) \leq C(1,v) = v$ (all $0 \leq u,v \leq 1$), and so $C(u,v) \leq \min(u,v) = M(u,v)$. The copula $M(u,v)$ is called the Frechet upper bound and can be interpreted as the copula with maximum positive dependence. Furthermore, $C(u,v) \geq max(u + v{-}1, 0) = W(u,v)$(all $0 \leq u,v \leq 1$); $W(u,v)$ is the Frechet lower bound, the copula with maximum negative dependence. The copula families are comprehensive: that is, they include the Frechet upper and lower bounds. Further, the so-called independence copula, $\pi(u,v) = uv$.

It is useful to measure the coverage of a copula family in terms of a standard measure of dependence, such as Kendall's τ , since not all copulas are comprehensive. Note that because Kendall's τ is based on ranks and is therefore invariant to a strictly increasing transformation of the margins, its properties depend only on the copula of the bivariate distribution. Furthermore $\tau \in [-1, 1]$ and $\tau = -1$ for W, $\tau = +1$ for M , and $\tau = 0$ for π (the Frechet lower and upper bounds, and independence copula).

There are a large number of Archimedian copula families (NELSEN, 2006). The choice of a copula family can be guided by the dependence properties of that family. Some properties are as follows. Copulas may be reflection symmetry—if specification of the joint survival function in terms of the copula gives rise to the same distribution as specification in terms of the distribution function, then the copula is reflection symmetric. Copulas may be: comprehensive or otherwise; extendable to more than 2 dimensions. They may also exhibit: varying degrees of upper and lower tail dependence; only negative or positive dependence structure. Copula families may be specified by more than one parameter in order to model dependence structure in more detail. NELSEN (2006) and JOE (1997) discuss a number of Archimedian copulas generated by multi-parameter Laplace transform families.

3 Main result

3.1 Sklar Theorem of Probability Measures on Product Spaces

Throughout this section, let $\Omega = \Omega_1 \otimes \Omega_2$, $A = A_1 \times A_2$ and $U = U_1 \times U_2$.

By Scarsini's technical report on copula of probability measures on product spaces (1989), for $(\Omega, \mathcal{A}, \mu)$ being a probability space of a product $\Omega = \Omega_1 \otimes \Omega_2$ and $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, we first assume $\mathcal{A}_i$, for all i, being increasing classes of subsets of Ω_i containing $\emptyset$ and Ω_i.

A class $\mathcal{A}$ of subsets of Ω is an increasing class if it is linearly ordered by set inclusion:

$$\forall A, B \in \mathcal{A}, \quad \text{either} \quad A \subset B, B \subset A \text{or} A = B$$

Moreover,

$$P(\Omega_1 \times A_2) = P_2(A_2), \forall A_2 \in \mathcal{A}_2$$

$$P(A_1 \times \Omega_2) = P_1(A_1), \forall A_1 \in \mathcal{A}_1$$

such that P_1 and P_2 be probability measure on Ω_1 and Ω_2 respectively.

In this paper, there are two relations on $\Omega = \Omega_1 \otimes \Omega_2$ defined for elements of each Ω_i and denoted by $\prec_{\mathcal{A}_i}$ and $\sim_{\mathcal{A}_i}$ which are symmetric complement of each other. The relations are defined as follow:

For $i = 1, 2$ any $x, y \in \Omega_i$:

$$x \prec_{\mathcal{A}_i} y \quad \text{iff} \quad \exists A \in \mathcal{A}_i, [x \in A \Longrightarrow y \notin A]$$

$$x \sim_{\mathcal{A}_i} y \quad \text{iff} \quad \forall A \in \mathcal{A}_i, [x \in A \Longleftrightarrow y \in A]$$

By the definition, we have that $\prec_{\mathcal{A}_i}$, simply $\prec$, is a weak order and $\sim_{\mathcal{A}_i}$, simply $\sim$ is an equivalence relation. Let $x \precsim y$ if either $x \prec y$ or $x \sim y$.

By Scarsini relating to Sklar's theorem, we apply to bivariate case as follow.

***Theorem* 3.1.** *There exists an unique subcopula $\tilde{C}_\mu^{\mathcal{A}\times\mathcal{B}}$ defined on $P_1(\mathcal{A}_1) \times P_2(\mathcal{A}_2)$ such that for all $A_i \in \mathcal{A}_i, i = 1, 2$*

$$\mu(A_1 \times A_2) = \tilde{C}_\mu^{\mathcal{A}\times\mathcal{B}}(P_1(\mathcal{A}_1) \times P_2(\mathcal{A}_2))$$

Any subcopula can be extended to a copula in more than one way. In order to generate an unique copula, it is obvious that nonatomicity of P_i for all i is necessary. The hypothesis that $P_1(\mathcal{A}_1) = P_2(\mathcal{A}_2) = I$ for $i = 1, 2$ is a sufficient condition but not necessary to be nonatomic. According to Scarsini, even with this condition, we will not be able to obtain a result analogous to general properties of copula in case of distribution function on $\mathbb{R}^2$ which are provided by the following theorems.

***Theorem* 3.2.** *For $i = 1, 2$, let $P_i(\mathcal{A}_i) = I$. Let Y_i be a polish space weakly ordered by $\prec$, and $\mathcal{B}_i$ is an increasing family of subsets of Y_i. Let $\mathcal{Y}_i$ be the Borel σ-field of subsets of Y_i. Let $f_i : X_i \to Y_i$ be a strongly monotone bijection with respect to $\prec_{\mathcal{A}_i}, \prec_{\mathcal{B}_i}$. Consider the space $(Y_1 \times Y_2, \mathcal{Y}_1 \otimes \mathcal{Y}_2, \mu f^{-1})$, where $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then $C_\mu^{\mathcal{A}} = C_{\mu f^{-1}}^{\mathcal{B}}$, by $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$.*

3.2 Capacity Functional

In the sense of real problems, a σ-field $\mathcal{A}$ of a random set X might not in the form of an increasing set.

Note that distribution functions, as set functions, are not additive in general. In fact,

$$P(X \subseteq A) + P(X \cap A^c \neq \emptyset) = 1, \forall A \subseteq U$$

Many applications require the use of set function that are not finitely additive, for instance, the characteristic function, which is monotone but not additive. The integration which is performed with respect to nonadditive probability instead of a Riemann or Lebesque integral is of Choquet. The main objective of Scarsini's notes on Distributions with Fixed Marginals and related topics (1996) is to study some properties

of capacities on a finite dimensional space. Scarsini examined the possibility of extending the concept of copula to more general situation of measure after defining the distribution function of a capacity. He used the assumption of convexity of a capacity to establish the existence of a function that links the multivariate distribution functions to its marginals called a generalized copula of the capacity. Moreover, this copula with d-monotone property has all the usual properties of a copula. Unlike the case of a σ-additive case, it is not necessary to know the value of a capacity on the whole Borel class, but only on a suitable subclass.
The definition of capacity in this paper differ from the original definition from Choquet(1953).

By the dual concept of a probability law of a probability space , we define $T : 2^U \to I$ by;

$$T(A) = P(X \cap A \neq \emptyset) = P(\omega : X(\omega) \in A)$$

A set function $T : 2^U \to I$ is a capacity functional of some random sets if it satisfies

(α) $T(\emptyset) = 0$ and $T(U) = 1$,

(β) For any $k \geq 2$ and $A_1, A_2, ..., A_k \in 2^U$,

$$T(\bigcap_{j=1}^{k} A_j) \leq \sum_{\emptyset \neq I \subseteq 1,2,...,k} (-1)^{|I|+1} T(\bigcup_{i \in I} A_i)$$

This also gives the definition of a dual $\tilde{T}(A)$ of a capacity T defined as $\tilde{T} = 1 - T(A^c)$ and $\tilde{T}$ is also a capacity function.

A capacity function T is then called 2-monotone; i.e., $\forall A, B \subseteq U$

$$T(A) + T(B) \leq T(A \cap B) + T(A \cup B)$$

We see that, by the definition, T is then convex. But, 2-monotonicity implies monotonicity in each argument only if T is grounded, satisfying the boundary condition.

3.2.1 Capacity Functional for Random Closed sets

If we consider random sets taking values in a discrete subset $\mathbb{D}$ of 2^U , then their probability laws are determined by their density f on $\mathbb{D}$, i.e., $\forall \mathbb{A} \subseteq 2^U$,

$$P(X \in \mathbb{A}) = \sum_{A \in \mathbb{D} \cap \mathbb{A}} f(A)$$

such that $f(A) = P(S = A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} F(B)$

If we view a finite set U as a topological space with its discrete topology, then a finite random set X is a random closed set, and its probability law on the σ-field of 2^U is uniquely determined by its capacity functional. So, a capacity T characterizes a probability measure Q on 2^U via $T(A) = Q(\mathcal{F}_A)$.

More specifically, there exist a space $(\Omega, \mathcal{A}, P)$, for a random element $S : \Omega \to 2^U$, such that, for all $A \subseteq U$,

$$P(S \cap A \neq \emptyset) = P_S(\mathcal{F}_A) = Q(\mathcal{F}_A) = T(A)$$

The space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$ of closed subsets of U and its Borel σ-field with the hit-or-miss topology turns out to be a metric space; compact, Hausdorff and second countable. Moreover, probability measure on their Borel σ-field are determined by their values on compact sets. For instance, if we let $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$ being the space of closed subsets of $\mathbb{R}^d$ and $S : (\Omega, \mathcal{A}, P) \to (\mathcal{F}(\mathbb{R}^d), \mathcal{B}(\mathcal{F}))$ being a random closed set on $\mathbb{R}^d$, the probability law on S is P_S on $\mathcal{B}(\mathcal{F})$ where $P_S = PS^{-1}$. The definition of T for S is the following.

Definition 3.3. A set function $T : K \to I$ is a capacity functional if it satisfies:

CF1 $0 \leq T \leq 1$ and $T(\emptyset) = 0$.

CF2 T is alternating of infinite order,i.e, for any $n \geq 2$ and $K_1, K_2, ..., K_n \in \mathcal{K}$,

$$T(\bigcap_{j=1}^{n} K_j) \leq \sum_{\emptyset \neq I \subseteq \{1,2,...,n\}} (-1)^{|I|+1} T(\bigcup_{i \in I} K_i)$$

CF3 If $K_n \searrow K$, then $T(K_n) \searrow T(K)$.

For random closed sets S on $\mathbb{R}^d$, we can consider T as

$$T(K) = P(S \cap K \neq \emptyset) = P(\mathcal{F}_K)$$

It turns out that T can characterize the probability measure on $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$ as the counterpart of Lebesgue-Stieltjes theorem as the following.

***Theorem* 3.4** (Choquet Theorem)*. If $T : \mathcal{K} \to I$ is a capacity functional, then there exists a unique probability measure P on $\mathcal{B}(\mathcal{F})$ such that for all $K \in \mathcal{K}$,*

$$P(\mathcal{F}_K) = T(K)$$

3.3 Distributions of Random Sets

Let S be a random set of a probability space $(\omega, \mathcal{A}, P)$which takes value on $\mathcal{F}$ being a class of closed subsets of U, that is a map $S : \Omega \to F$ defined by

$$S^{-}(K) = \{\omega \in \Omega : S(\omega) \cap K \neq \emptyset\} = S^{-1}(\mathcal{F}_K) \in \mathcal{A}$$

for all $K \in \mathcal{B}(\mathcal{F})$. The so-called Effros-$\sigma$-algebra $\mathcal{B}(\mathcal{F})$ is generated by $\mathcal{F}_{K\,K\in\mathcal{K}}$, which $\mathcal{K}_K = F \in \mathcal{F} : F \cap K \neq \emptyset$, and is the Borel-$\sigma$-field with respect to the fell topology. In addition, the distribution functions of S is then the image measure P_S of P on $\mathcal{B}(\mathcal{F})$.

3.3.1 Joint Distributions of Random Sets

Let $S = (S_1, S_2)$ be a random set of a probability space $(\Omega, \mathcal{A}, P)$,which $\Omega = \Omega_1 \otimes \Omega_2$, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ and takes value on closed subsets of $U = U_1 \times U_2$. H is a joint distribution function of random sets (S_1, S_2) such that $H : 2^U \to I$ defined by

$$H(A_1, A_2) = P(S_1 \subseteq A_1, S_2 \subseteq A_2)$$

$$= P(\{\omega = (\omega_1, \omega_2) : S_1(\omega_1) \subseteq A_1, S_2(\omega_2) \subseteq A_2\})$$

which satisfies all conditions of joint distribution functions of random sets which are grounded, marginal and 2-monotone.

From the one dimensional case, the product-Effros-σ-field $\mathcal{B}(\mathcal{F}_1 \times \mathcal{F}_2)$ generated by $\mathcal{F}_{K_1} \times \mathcal{F}_{K_2}$ can be inferred where $K_i \in \mathcal{K}_i$. Their joint probability distribution is then given by $P(\{(\omega_1, \omega_2) : S_1(\omega_1) \times S_2(\omega_2) \cap K_1 \times K_2 \neq \emptyset\}) = P(S_1^{-}(K_1) \cap S_2^{-}(K_2))$. From now on, a set function satisfying the following conditions shall be called multivariate capacity functional.

Definition 3.5. Let $\varphi : \mathcal{K}_1 \times \mathcal{K}_2 \to I$ satisfying the following conditions:

$(\mathbf{MCF1})$ $\varphi(\emptyset \times K_2) = 0$ and $\varphi(K_1 \times \emptyset) = 0$

$(\mathbf{MCF2})$ For all $j = 1, 2$ and $K = (K_1, K_2)$, $K^j = (K_1^j, K_2^j) \in K$, it holds that

$$\Delta_2\varphi(K; K^1, K^2) \geq 0$$

where $\Delta_0\varphi(K) = 1 - \varphi(K_1 \times K_2)$, $\Delta_2\varphi(K; K^1, K^2) = \Delta_1\varphi(K; K^1) - \Delta_1\varphi(K \cup K^2; K^1)$.

$(\mathbf{MCF3})$ For all decreasing sequences $\{K_i^k\}_{k=1}^{\infty} \subseteq \mathcal{K}_i$ for $i = 1, 2$, it holds that $\varphi(K_1^k \times K_2^k) \searrow \varphi(K_1 \times K_2)$ where $K_i = \bigcap_{k=1}^{\infty} K_i^k$.
Then, φ is a bivariate capacity functional of (S_1, S_2).

***Theorem* 3.6.** *φ defined by $\varphi(K_1 \times K_2) = P_{S_1 \times S_2}(\mathcal{F}_{K_1} \times \mathcal{F}_{K_2})$ is the (bivariate) capacity functional of $S = (S_1, S_2)$.*

3.4 Copulas for Random Closed Sets

An emerging literature with in this report, the idea was developed as the following.

For each $i = 1, 2$, let S_i be a random set taking value on closed subsets of U_i of a probability space $(\Omega_i, \mathcal{A}_i, P_i)$. For $i = 1, 2$, $\mathbb{A}_i$ is a σ-field which is a subclass of the power set of 2^{U_i} and F_i is a distribution function in each S_i.

By applying the idea from Scarsini, the following lemma has been developed.

***Lemma* 3.7.** *Let $A_i, B_i \in \mathcal{K}_i$ for $i = 1, 2, ..., d$. If φ is d-monotone, then C defined as $C(T_1(K_1), ..., T_d(K_d)) = \varphi(\times_{i=1}^{d} K_i)$ is n-increasing for all $n \leq d$.*

Proof. Without any lost of generality, we will prove that C is increasing in the first n dimensions. Let $C_j = B_1 \times B_2 \times ... \times A_j \times ... \times B_d$ and let $E_1, ..., E_n$ be disjoint sets in $\times_{i=1}^{d} \mathcal{K}_i$ such that

$$\bigcup_{j=1}^{n} E_j = \times_{j=1}^{n} (B_j \setminus A_j) \times_{j=n+1}^{d} B_j$$

Define $D_i = C_i \cup E_i$ for $i = 1, ..., n$.
Then, $D_i \cap D_j = C_i \cap C_j$ and $\bigcup_{j=1}^{n} D_j = \times_{j=1}^{d} B_j$.
Since φ is d-monotone,

$$\begin{aligned} \varphi(\bigcup_{j=1}^{n} D_j) &\geq \sum_{\emptyset \neq I \subseteq 1,...,n} (-1)^{|I|+1} \varphi(\bigcap_{i \in I} D_i) \\ &= \sum_{\emptyset \neq I \subseteq 1,...,n} (-1)^{|I|+1} \varphi(\bigcap_{i \in I} C_i) \end{aligned}$$

Thus,

$$\varphi(\times_{j=1}^{d} B_j) \geq \sum_{W_j = A_j, B_j; j=1,...,n} (-1)^{\#(W_j = A_j)+1} \varphi(\times_{j=1}^{d} W_j)$$

So,

$$\varphi(\times_{j=1}^{d} B_j) + \sum_{W_j = A_j, B_j; j=1,...,n} (-1)^{\#(W_j = A_j)} \varphi(\times_{j=1}^{d} W_j) \geq 0$$

We have that

$$\begin{aligned} 0 &\leq (-1)^{\#(W_j=A_j;j=1,\dots,n)}C(T_1(W_1),...,T_n(W_n), \\ &, \quad T_{n+1}(B_{n+1}),...,T_d(B_d)) \\ &= (-1)^{\#(w_j=u_j)}C(w_1,...,w_n,v_{n+1},...,v_d) \end{aligned}$$

It means that C is n-increasing.

***Theorem* 3.8.** *Let φ be a Choquet-capacity functional on $K = (K_1, K_2)$ and each K_i of which T_i is its capacity functional. Then there exists a unique sub-copula C such that*

(1) $Dom(C) = Ran(T_1) \times Ran(T_2)$,

(2) *For all* $(K_1, K_2) \in \mathcal{K}$,

$$C(T_1(K_1), T_2(K_2)) = \varphi(K_1 \times K_2)$$

Proof. 1) C is well-defined.
Since $T_i(\emptyset) = 0$ and $P(\mathcal{F}) = P(\mathcal{F}^{\emptyset}) = 1 - P(\mathcal{F}_{\emptyset}) = 1 - T_i(\emptyset)1$, we have $0, 1 \in RanT_i$.

Let $\alpha_i, \beta_i \in RanT_i$ and $\alpha_i = \beta_i$ for $i = 1, 2$. There exist $A_i, B_i \in K_i$ such that $T_i(A_i) = \alpha_i$ and $\beta_i = T_i(B_i)$. Then, for $i = 1, 2$,

$$T_i(A_i) = T_i(B_i)$$

$$|\varphi(B_1 \times B_2) - \varphi(A_1 \times A_2)|$$

$$\leq |T_1(B_1) - T_1(A_1)| + |T_2(B_2) - T_2(A_2)| = 0$$

Then, $\varphi(B_1 \times B_2) = \varphi(A_1 \times A_2)$.

$$C(T_1(A_1), T_2(A_2)) = C(T_1(B_1), T_2(B_2))$$

Thus, $C(\alpha_1, \alpha_2) = C(\beta_1, \beta_2)$.

2) C is grounded.
suppose $T_1(K_1) = 0$, that is

$$P((\omega_1, \omega_2) : S_1(\omega_1) \cap K_1 \neq \emptyset) = 0$$

. Thus, $C(T_1(K_1), T_2(K_2))$
$= \varphi(K_1 \times K_2)$
$= P_S(\mathcal{F}_{K_1} \times \mathcal{F}_{K_2})$
$= P((\omega_1, \omega_2) : S_1(\omega_1) \cap K_1 \neq \emptyset, S_2(\omega) \cap K_2 \neq \emptyset)$
$= 0$

Also, $C(T_1(K_1), T_2(K_2)) = 0$ for $T_2(K_2) = 0$. Thus, $C(0, u_2) = 0$ and $C(u_1, 0) = 0$.

3) C is marginal.
Since S_i is a random closed set taking value on non-empty closed subsets of U_i,wehave $T_i(U_i) = P(\omega_i : S_i(\omega) \cap U_i \neq \emptyset) = 1$.
$C(T_1(U_1), T_2(K_2))$
$= \varphi(U_1 \times K_2)$
$= P((\omega_1, \omega_2) : S_1(\omega) \cap U_1 \neq \emptyset, S_2(\omega) \cap K_2 \neq \emptyset)$
$= P(\omega_2 : S_2(\omega) \cap K_2 \neq \emptyset)$
$= T_2(K_2)$
Also,$C(T_1(K_1), T_2(K_2)) = T_1(K_1)$ for $T_2(K_2) = 1$.
Thus, $C(1, u_2) = u_2$ and $C(u_1, 1) = u_1$.

4) C is 2-increasing.
Let $A_i, B_i \in K_i$. By the previous lemma, C is n-increasing since φ is 2- monotone, for $n = 1, 2$.

3.5 Quasi-inverses of Distribution Functions

Definition 3.9. For generating sets $\mathcal{F}_K$ of $\mathcal{B}(\mathcal{F})$ in one dimension, let $T : K \to I$ such that $T(K) = P_S(\mathcal{F}_K)$. Then, the quasi-inverse T^{-1} with domain I is defined by

(1) if $t \in Ran(T)$, then there is $K = T^{-1}(t) \in K$ such that

$$T(T^{-1}(t)) = t$$

(2) if $t \notin Ran(T)$, then

$$T^{-1}(t) = \bigcap_{K_\alpha \in \mathcal{K}, P(\mathcal{F}_K) \geq t} K_\alpha$$

***Theorem* 3.10.** *Let φ, T_1, T_2 be as in the above theorem. Then, for any $u_1, u_2 \in [0, 1]$, there exists a sub-copula C such that,*

$$C(u_1, u_2) = \varphi(T_1^{-1}(u_1) \times T_2^{-1}(u_2))$$

C is uniquely determined on $Ran(T_1) \times Ran(T_2)$.

4 Conclusion

In this research, we have the theorems to construction a unique sub-copula C which we can apply to any other random closed set, for example, random (vertice) graph.

***Lemma* 4.1.** *Let $A_i, B_i \in \mathcal{K}_i$ for $i = 1, 2, ..., d.$ If φ is d-monotone, then C defined as $C(T_1(K_1), ..., T_d(K_d)) = \varphi(\times_{i=1}^{d} K_i)$ is n-increasing for all $n \leq d.$*

***Theorem* 4.2.** *Let φ be a Choquet-capacity functional on $K = (K_1, K_2)$ and each K_i of which T_i is its capacity functional. Then there exists a unique sub-copula C such that*

(1) $Dom(C) = Ran(T_1) \times Ran(T_2)$,

(2) *For all* $(K_1, K_2) \in \mathcal{K}$,

$$C(T_1(K_1), T_2(K_2)) = \varphi(K_1 \times K_2)$$

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